

**ECE 158B, Data Networks II, Spring Quarter, 1999**  
**Solutions of H.W. set 3**

1. First we will compute  $R_{in}(t)$  from  $r_{in}(t)$ . By definition,  $R_{in}(t) = \int_{-\infty}^t r_{in}(\tau)d\tau$ . Then  $R_{out}(t)$  is computed as<sup>1</sup>  $R_{out}(t) = R_{in} * \mu_C(t)$ , where (assuming  $C = 1$ )

$$\mu_C(t) = \begin{cases} Ct & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad (1)$$

By definition,  $B(t) = R_{in}(t) - R_{out}(t)$ . Finally, the virtual delay  $D(t)$  is defined by:

$$D(t) = \inf\{\Delta : \Delta \geq 0 \text{ and } R_{out}(t + \Delta) \geq R_{in}(t)\} \quad (2)$$

To understand equation (2) and to see why  $D(t)$  is called *virtual*, see the example depicted in Figure 1. For the case of *fixed-rate* servers,  $D(t)$  can be easily obtained from  $B(t)$  instead of evaluating equation (2). The next claim 1 states and proves this result.

**CLAIM 1** (Relationship between  $D(t)$  and  $B(t)$  of fixed-rate servers) *For fixed-rate servers of capacity  $C$  bits/sec ( $C > 0$ ), there holds*

$$D(t) = \frac{1}{C}B(t) \quad (3)$$

**Proof:** The key observation is that for small enough  $\epsilon > 0$ ,  $R_{out}(t)$  for fixed-rate servers of capacity  $C$  bits/sec can be written as follows:

$$R_{out}(t) = \min\{R_{in}(t), R_{out}(t^* - \epsilon) + C(t - t^*)\} \quad (4)$$

where

$$t^* = \sup\{\tau : \tau \leq t \text{ and } R_{out}(\tau) = R_{in}(\tau)\} \quad (5)$$

Roughly speaking,  $t^*$  is the *last* time  $R_{out} = R_{in}$  or equivalently the last time the buffer was empty. Equation (4) implies that either  $R_{out}(t) = R_{in}(t)$  or  $R_{out}(t) = R_{out}(t^* - \epsilon) + C(t - t^*)$ . If  $R_{out}(t) = R_{in}(t)$ , then  $B(t) = 0$  and  $D(t) = \inf\{\Delta : \Delta \geq 0 \text{ and } R_{out}(t + \Delta) \geq R_{in}(t)\} = 0$ . Otherwise,  $R_{out}(t) = R_{out}(t^* - \epsilon) + C(t - t^*)$ , hence  $B(t) = R_{in}(t) - R_{out}(t) = R_{in}(t) - R_{out}(t^* - \epsilon) - C(t - t^*)$ . And in this case we have by equation (2):

$$\begin{aligned} D(t) &= \inf\{\Delta : \Delta \geq 0 \text{ and } R_{out}(t^* - \epsilon) + C(t + \Delta - t^*) \geq R_{in}(t)\} \\ &= \inf\{\Delta : \Delta \geq 0 \text{ and } \Delta \geq \frac{1}{C}[R_{in}(t) - R_{out}(t^* - \epsilon) - C(t - t^*)]\} \\ &= \inf\{\Delta : \Delta \geq 0 \text{ and } \Delta \geq \frac{1}{C}[B(t)]\} \\ &= \frac{1}{C}B(t) \end{aligned}$$

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<sup>1</sup>This equation holds only when  $S = \bar{S} = \mu_C(t)$ . Alternatively, for the special case of fixed-rate servers of capacity  $C$  bits/sec,  $R_{out}(t)$  can be computed in a simple way. Note that  $R_{out}(t)$  results from the server leaking at a fixed-rate of  $C$  bits/sec *whenever* the buffer is not empty, i.e., whenever  $B(t) > 0$  (and at zero rate otherwise). A simple formula for  $R_{out}(t)$  can be obtained by taking into account the fact that  $R_{out}(t)$  is a non-decreasing right-continuous function. See the proof of claim 1, equation (4), for such a formula.

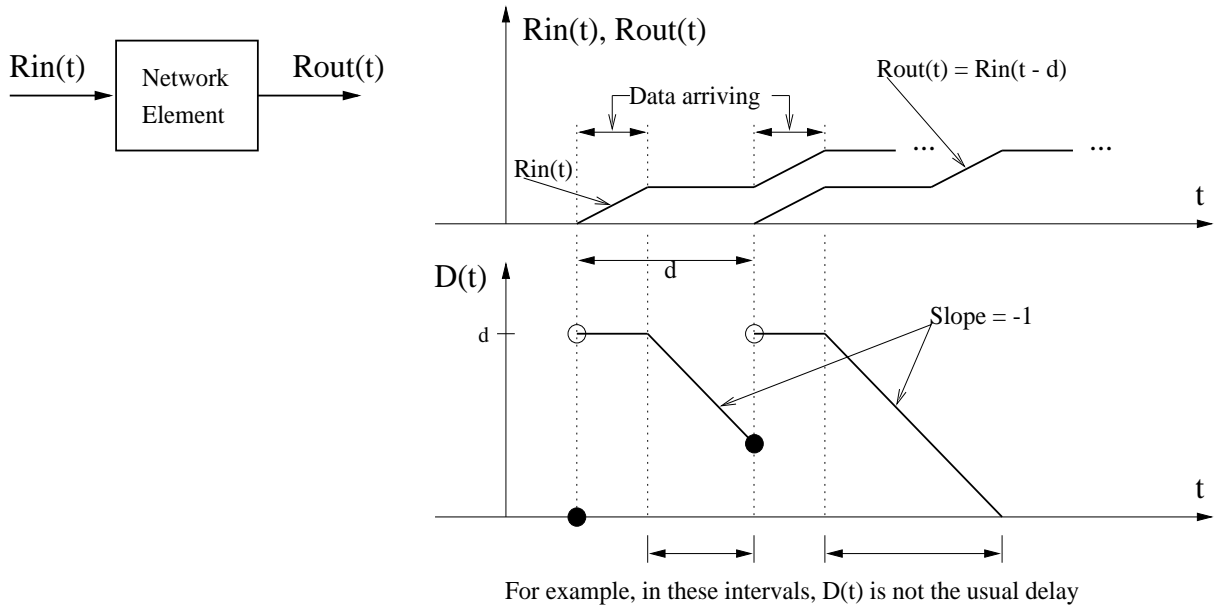


Figure 1: An example to explain the virtual delay  $D(t)$  of equation (2). We assume a known process  $R_{in}$  to be the input process to a *network element*. The network element is assumed to delay each data bit by exactly  $d$  seconds, where  $d$  is a constant. Hence  $R_{out}(t) = R_{in}(t - d)$ , and therefore we know that every *real* data bit that arrives at time  $t$  has a delay  $D(t) = d$  seconds (assuming data bits depart the network element in the same order in which they arrive). Equation (2) yields the actual delay during the intervals in which data arrives, i.e., equation (2) gives  $D(t) = d$  during intervals in which data arrives. However,  $D(t)$  of equation (2) is well defined  $\forall t$  including the intervals in which no data is arriving at the system. Clearly,  $D(t)$  during these intervals does not represent *actual* delay. Moreover  $D(t)$  of equation (2) is also well defined even if data does not depart the network element in the same order in which it arrives (in this case equation (2) does not yield the “usual delay” even during the intervals in which data arrives) and even if the arrival stream is not causal. Therefore,  $D(t)$ , in general, is called the *virtual* delay.

The last equality resulted since  $B(t)$  is by definition right-continuous and  $B(t) \geq 0 \forall t$ . This completes the proof.  $\blacksquare$

Define

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \quad (6)$$

(a) See Figure 2.

$$R_{in}(t) = u(t)$$

$$R_{out}(t) = t[u(t) - u(t - 1)] + u(t - 1)$$

$$B(t) = D(t) = (1 - t)[u(t) - u(t - 1)]$$

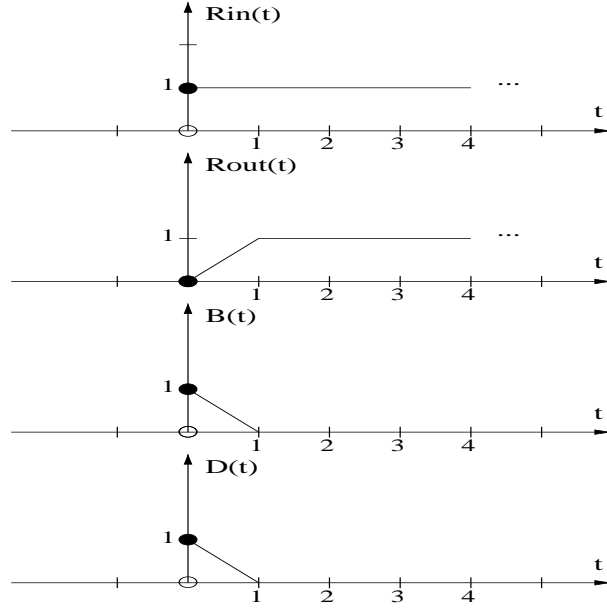


Figure 2:  $R_{in}(t)$ ,  $R_{out}(t)$ ,  $B(t)$  and  $D(t)$ , respectively, for problem 1, part (a).

(b) See Figure 3.

$$R_{in}(t) = u(t) + u(t - 2)$$

$$R_{out}(t) = t[u(t) - u(t - 1)] + u(t - 1) + (t - 2)[u(t - 2) - u(t - 3)] + u(t - 3)$$

$$B(t) = D(t) = (1 - t)[u(t) - u(t - 1)] + (3 - t)[u(t - 2) - u(t - 3)]$$

(c) See Figure 4.

$$R_{in}(t) = u(t) + u\left(t - \frac{1}{2}\right)$$

$$R_{out}(t) = t[u(t) - u(t - 2)] + 2u(t - 2)$$

$$B(t) = D(t) = (1 - t)\left[u(t) - u\left(t - \frac{1}{2}\right)\right] + (2 - t)\left[u\left(t - \frac{1}{2}\right) - u(t - 2)\right]$$

2. Let us assume  $r(t)$  to be causal signal, i.e., let

$$r(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sum_{k=0}^{\infty} [u(t - kT) - u(t - 1 - kT)] & \text{if } t \geq 0 \end{cases} \quad (7)$$

where  $T = 10$ . Hence,  $R(t) = \int_{-\infty}^t r(\tau)d\tau = \int_0^t r(\tau)d\tau$ . For  $E(t)$  to be an envelope of  $R(t)$ , we must have (by definition):  $R(t) - R(s) \leq E(t - s) \quad \forall s \leq t$ . Fix  $t$  and let  $s \leq t$

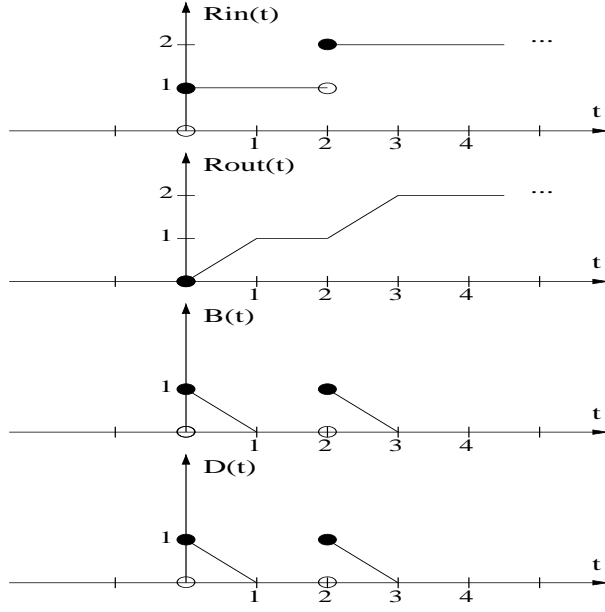


Figure 3:  $R_{in}(t)$ ,  $R_{out}(t)$ ,  $B(t)$  and  $D(t)$ , respectively, for problem 1, part (b).

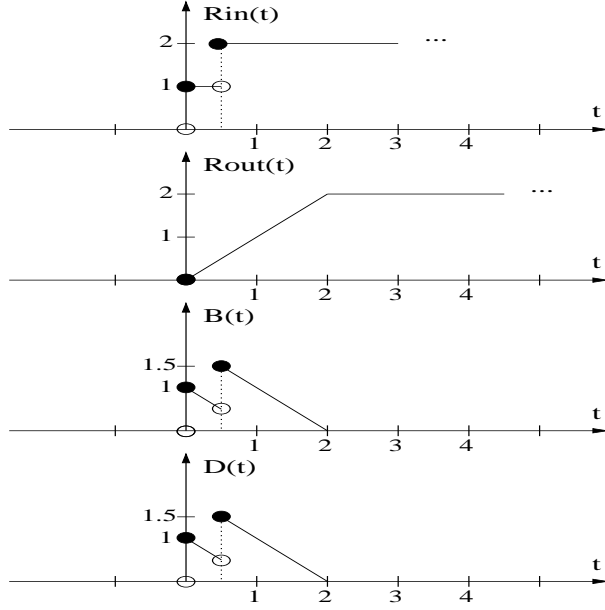


Figure 4:  $R_{in}(t)$ ,  $R_{out}(t)$ ,  $B(t)$  and  $D(t)$ , respectively, for problem 1, part (c).

be arbitrary. Note that if  $t < 0$ , then  $R(t) - R(s) = \int_{-\infty}^t r(\tau)d\tau - \int_{-\infty}^s r(\tau)d\tau = 0$  since  $r(\tau) = 0$  for  $\tau < 0$ . On the other hand, if  $t \geq 0$ , then

$$R(t) - R(s) = \int_{-\infty}^t r(\tau)d\tau - \int_{-\infty}^s r(\tau)d\tau = \int_s^t r(\tau)d\tau \leq \int_0^t r(\tau)d\tau$$

with equality if  $s \leq 0$ . Hence,  $\forall t \geq 0$ :  $E(t) = \int_0^t r(\tau)d\tau$ , and from (7) we get

$$\forall t \geq 0: E(t) = \sum_{k=0}^{\infty} [(t - kT)u(t - kT) - (t - 1 - kT)u(t - 1 - kT)]$$

From the above,  $E(t) \forall t$  is given by:

$$E(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sum_{k=0}^{\infty} [(t - kT)u(t - kT) - (t - 1 - kT)u(t - 1 - kT)] & \text{if } t \geq 0 \end{cases} \quad (8)$$

$r(t)$  and  $E(t)$  are shown (not to scale) in Figure 5.

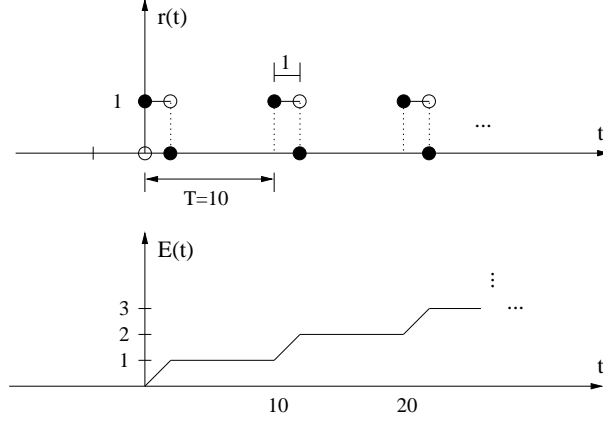


Figure 5:  $r(t)$  and  $E(t)$ , respectively.

3. (a) The backlog  $B(t)$  is such that  $B(t) \leq b_{max}$ , where

$$b_{max} = \sup_{\tau \in \mathbb{R}} \{E_{in}(\tau) - S(\tau)\} \quad (9)$$

Note that since  $E_{in}(\tau) = S(\tau) = 0 \forall \tau < 0$ , equation (9) can be rewritten as

$$\begin{aligned} b_{max} &= \max \left\{ \sup_{\tau < 0} \{E_{in}(\tau) - S(\tau)\}, \sup_{\tau \geq 0} \{E_{in}(\tau) - S(\tau)\} \right\} \\ &= \max \left\{ 0, \sup_{\tau \geq 0} \{E_{in}(\tau) - S(\tau)\} \right\} \end{aligned} \quad (10)$$

Now we turn to evaluate  $\sup_{\tau \geq 0} \{E_{in}(\tau) - S(\tau)\}$ . Assuming  $\sigma \geq 0$  and  $0 < \rho \leq C$  (as given), we consider two cases: Either  $\rho = C$  or  $\rho \neq C$  (i.e.,  $0 < \rho < C$ ):

If  $\rho = C$ ,  $E_{in}(x) = \min\{Cx, \sigma + \rho x\} = Cx \quad \forall x \geq 0$ . This implies

$$\begin{aligned} \sup_{\tau \geq 0} \{E_{in}(\tau) - S(\tau)\} &= \sup_{\tau \geq 0} \{C\tau - \rho(\tau - T)^+\} \\ &= CT = \rho T \end{aligned} \quad (11)$$

If  $\rho \neq C$ , we need to consider two subcases (with the help of Figure 6 (c), where for  $\rho \neq C$ ,  $e^* = C \frac{\sigma}{C-\rho}$ ,  $x^* = \frac{\sigma}{C-\rho}$  and  $x^{**} = \frac{e^* + \rho T}{\rho}$ ): Either  $T \geq x^*$  or  $T < x^*$ . If  $T \geq x^*$ , then

$$\begin{aligned} \sup_{\tau \geq 0} \{E_{in}(\tau) - S(\tau)\} &= \sup_{\tau \geq 0} \{\min\{C\tau, \sigma + \rho\tau\} - \rho(\tau - T)^+\} \\ &= \min\{CT, \sigma + \rho T\} = \sigma + \rho T \end{aligned} \quad (12)$$

where the last equality resulted since  $T \geq x^* = C \frac{\sigma}{C-\rho}$ . If  $T < x^*$ , then

$$\begin{aligned}
\sup_{\tau \geq 0} \{E_{in}(\tau) - S(\tau)\} &= \sup_{\tau \geq 0} \{\min\{C\tau, \sigma + \rho\tau\} - \rho(\tau - T)^+\} \\
&= \min\{Cx^*, \sigma + \rho x^*\} - \rho(x^* - T)^+ \\
&= \sigma + \rho x^* - \rho(x^* - T)^+ \\
&= \sigma + \rho T
\end{aligned} \tag{13}$$

From (11), (12), (13) and (10) we have

$$b_{max} = \begin{cases} CT & \text{if } \rho = C \\ \sigma + \rho T & \text{if } 0 < \rho < C \end{cases} \tag{14}$$

(b) Similarly for  $D(t)$ , the virtual delay  $D(t)$  is such that  $D(t) \leq d_{max}$ , where

$$d_{max} = \inf\{d : d \geq 0 \text{ and } S \geq E_{in} * \delta_d\} \tag{15}$$

Now, if  $\rho = C$ , then trivially  $d_{max} = T$ . If  $0 < \rho < C$ , then from Figure 6 (c) (and based on equation (15))  $d_{max} = x^{**} - x^* = T + \frac{\sigma}{\rho}$ . Therefore,

$$d_{max} = \begin{cases} T & \text{if } \rho = C \\ T + \frac{\sigma}{\rho} & \text{if } 0 < \rho < C \end{cases} \tag{16}$$

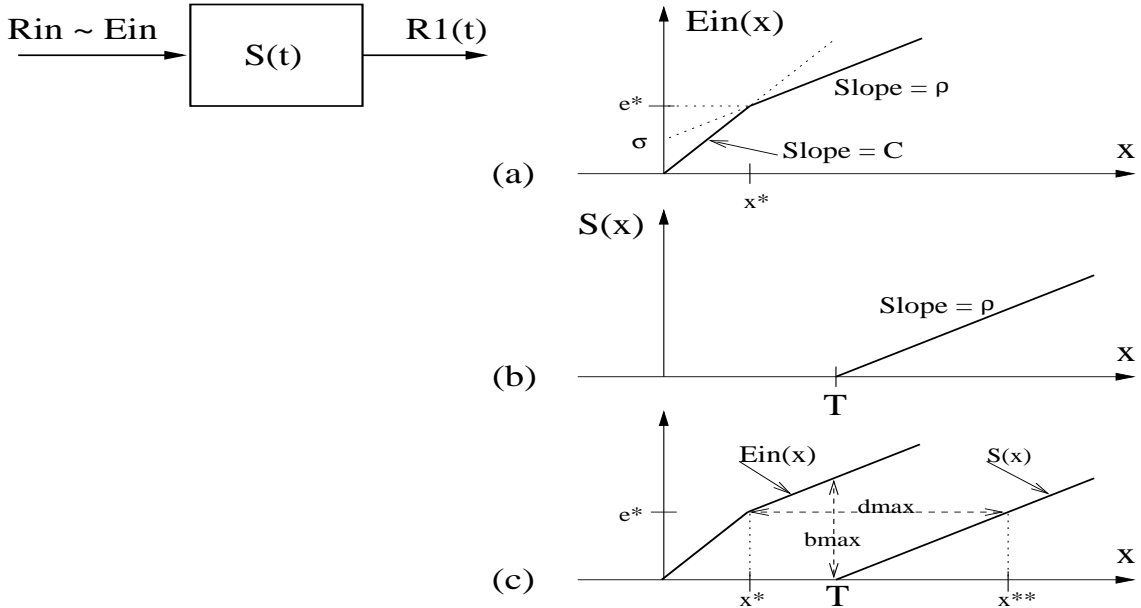


Figure 6:  $E_{in}(x)$  and  $S(x)$  in (a) and (b), respectively, for  $\rho \neq C$ . Part (c) shows a graphical interpretation of the maximum backlog of the *first* network element,  $b_{max}$ , as the maximum *vertical* distance between  $E_{in}(x)$  and  $S(x)$  and the maximum delay of the *first* network element,  $d_{max}$ , as the maximum *horizontal* distance between  $E_{in}(x)$  and  $S(x)$ . Assuming  $\rho \neq C$ :  $x^* = \frac{\sigma}{C-\rho}$ ,  $e^* = C \frac{\sigma}{C-\rho}$  and  $x^{**} = \frac{e^* + \rho T}{\rho}$ .

(c) It was shown in class that  $E_{out} = (E_{in} * \bar{S}) \otimes S$ . Since  $\bar{S}$  is not known, let us assume that<sup>2</sup>  $\bar{S} = \delta(t)$  where  $\delta(t)$  is defined in class. Hence,  $E_{out} = E_{in} \otimes S$ . We need to compute  $E_{in} \otimes S$ . However, it is easier to compute  $E_{in} \otimes S'$  where  $S'(x) = S(x+T) = \rho(x)^+ = S * \delta_{-T}(x)$ . Once  $E_{in} \otimes S'(x)$  is found, we can use the following claim 2 to obtain  $E_{in} \otimes S(x)$  from  $E_{in} \otimes S'$ . Therefore, with the help of a graph<sup>3</sup> of  $E_{in}$  and  $S'$ , it is not hard to see that

$$\begin{aligned} E_{in} \otimes S'(x) &= \sup_{\tau \in \mathbb{R}} \{E_{in}(x+\tau) - S'(\tau)\} \\ &= \sup_{\tau \in \mathbb{R}} \{E_{in}(x+\tau) - \rho(\tau)^+\} \\ &= \begin{cases} (\rho x)^+ & \text{if } \rho = C \\ (\sigma + \rho x)^+ & \text{if } 0 < \rho < C \end{cases} \end{aligned} \quad (17)$$

But  $E_{in} \otimes S(x) = E_{in} \otimes (S' * \delta_T)$ , where  $E_{in} \otimes (S' * \delta_T)$  can be computed using  $E_{in} \otimes S'$  from (17) in the following way:

$$\begin{aligned} E_{in} \otimes (S' * \delta_T) &\stackrel{\text{by claim 2}}{\underset{\downarrow}{=}} (E_{in} \otimes S') * \delta_{-T} = \begin{cases} (\rho x)^+ * \delta_{-T} & \text{if } \rho = C \\ (\sigma + \rho x)^+ * \delta_{-T} & \text{if } 0 < \rho < C \end{cases} \\ &= \begin{cases} (\rho(x+T))^+ & \text{if } \rho = C \\ (\sigma + \rho(x+T))^+ & \text{if } 0 < \rho < C \end{cases} \end{aligned}$$

Therefore,

$$E_{out} = E_{in} \otimes S(x) = \begin{cases} (\rho(x+T))^+ & \text{if } \rho = C \\ (\sigma + \rho(x+T))^+ & \text{if } 0 < \rho < C \end{cases} \quad (18)$$

**CLAIM 2** (A useful property of deconvolution) *For any processes  $F$  and  $G$  and a real constant  $d$ , there holds*

$$F \otimes (G * \delta_d) = (F \otimes G) * \delta_{-d} \quad (19)$$

**Proof:** Since  $G * \delta_d(x) = G(x-d)$ , then by definition of the deconvolution,  $F \otimes (G * \delta_d) = \sup_{\tau \in \mathbb{R}} \{F(x+\tau) - G(\tau-d)\}$ . Letting  $\tau-d = \tau'$ , we get

$$\begin{aligned} F \otimes (G * \delta_d) &= \sup_{\tau \in \mathbb{R}} \{F(x+\tau) - G(\tau-d)\} = \sup_{\tau' \in \mathbb{R}} \{F(x+d+\tau') - G(\tau')\} \\ &= (F \otimes G)(x+d) \\ &= (F \otimes G) * \delta_{-d} \end{aligned}$$

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<sup>2</sup>Since  $R_{out} \leq R_{in}$  for any real network element, and since  $\delta(t)$  is the *identity* process, i.e.,  $R_{in} * \delta(t) = R_{in}(t)$ , it follows that  $R_{out} \leq R_{in} * \delta(t)$  and hence  $\delta(t)$  is an upper bound on  $\bar{S}$  of any real network element.

<sup>3</sup>For direct computation, consider cases of intervals of  $x$ . When  $\rho = C$ , note that  $E_{in}$  becomes  $S'$ , hence  $E_{in} \otimes S'(x)$  is easily computed to be  $S'(x) \otimes S'(x) = S'(x) = (\rho x)^+$ .

(d) First, we find upper bounds to  $B_2(x)$  and  $D_2(x)$  of the second network element. Note that  $E_{out}$  computed in part (c) above is now an envelope on the input process to the second network element. With this, upper bounds on  $B_2(x)$  and  $D_2(x)$  are computed in the same way of computing the upper bounds in parts (a) and (b), respectively. The only difference is replacing  $E_{in}$  of parts (a) and (b) by  $E_{out}$  of part (c). Therefore,

$$\begin{aligned}
B_2(x) &\leq \sup_{\tau \in \mathbb{R}} \{E_{out}(\tau) - S(\tau)\} \\
&= \begin{cases} \sup_{\tau \in \mathbb{R}} \{(\rho(\tau + T))^+ - \rho(\tau - T)^+\} & \text{if } \rho = C \\ \sup_{\tau \in \mathbb{R}} \{(\sigma + \rho(\tau + T))^+ - \rho(\tau - T)^+\} & \text{if } 0 < \rho < C \end{cases} \\
&= \begin{cases} \rho(2T) & \text{if } \rho = C \\ \sigma + \rho(2T) & \text{if } 0 < \rho < C \end{cases}
\end{aligned}$$

where the last equality resulted because  $\tau = T$  achieves the “sup”. Similarly for  $D_2(x)$ , we have

$$\begin{aligned}
D_2(x) &\leq \inf\{d : d \geq 0 \text{ and } S \geq E_{out} * \delta_d\} \\
&= \begin{cases} \inf\{d : d \geq 0 \text{ and } \rho(x - T)^+ \geq (\rho(x + T))^+ * \delta_d\} & \text{if } \rho = C \\ \inf\{d : d \geq 0 \text{ and } \rho(x - T)^+ \geq (\sigma + \rho(x + T))^+ * \delta_d\} & \text{if } 0 < \rho < C \end{cases} \\
&= \begin{cases} 2T & \text{if } \rho = C \\ 2T + \frac{\sigma}{\rho} & \text{if } 0 < \rho < C \end{cases}
\end{aligned}$$

where the last equality resulted since the *horizontal* distance between  $S(x)$  and  $E_{out}$  is fixed and equals  $2T$  if  $\rho = C$  and  $2T + \frac{\sigma}{\rho}$  if  $0 < \rho < C$ . See Figure 7 (b) for illustration when  $0 < \rho < C$ .

Second, we find an upper bound to the total virtual end-to-end delay  $D_{total}(x)$  for the two elements. See Figure 7 (a). The entire system of two network elements in series delivers the (minimum) service curve  $S_{total} = S * S$ . To find  $S_{total}$ , note that  $S(x) = \rho(x - T)^+ = S' * \delta_T$ , where  $S'(x) = S(x + T) = \rho(x)^+$ . Therefore,  $S_{total} = S * S = (S' * \delta_T) * (S' * \delta_T) = (S' * S') * (\delta_T * \delta_T) = S' * \delta_{2T}$ . The last equality resulted since  $S'$  is *causal concave* (in the sense defined in the class notes) process and hence  $S' * S' = S' \wedge S' = S'$ . Now we have  $S_{total} = S' * \delta_{2T} = \rho(x - 2T)^+$ . Since  $S_{total}$  has the same form of  $S$  with  $2T$  in  $S_{total}$  in place of the  $T$  in  $S$ , and since  $E_{in}$  (used in deriving equation (16) in part (b)) is an envelope on the input process to the entire system, then equation (16) holds for the entire system after replacing the  $T$  by  $2T$ , i.e.,

$$D_{total}(x) \leq d_{max} = \begin{cases} 2T & \text{if } \rho = C \\ 2T + \frac{\sigma}{\rho} & \text{if } 0 < \rho < C \end{cases} \quad (20)$$

Similarly for  $B_{total}(x)$ , based on (14) and the above,

$$B_{total}(x) \leq b_{max} = \begin{cases} C(2T) & \text{if } \rho = C \\ \sigma + \rho(2T) & \text{if } 0 < \rho < C \end{cases} \quad (21)$$

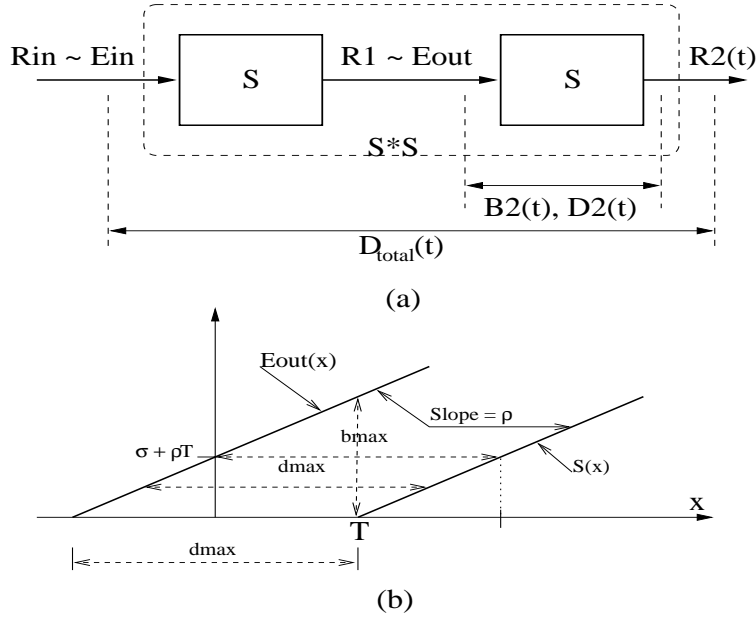


Figure 7: Schematic diagram of the system of two network elements in series, in (a). Part (b) shows a graphical interpretation (for  $\rho \neq C$ ) of the maximum backlog of the *second* network element,  $b_{max}$ , as the maximum *vertical* distance between  $E_{out}(x)$  and  $S(x)$  and the maximum delay of the *second* network element,  $d_{max}$ , as the maximum *horizontal* distance between  $E_{out}(x)$  and  $S(x)$ .

4.  $S_1(x)$  and  $S_2(x)$  can be rewritten as follows:

$$S_1(x) = 10(x - 1)^+ = (10x)^+ * \delta_1(x) \quad \text{and} \quad S_2(x) = 9(x - 2)^+ = (9x)^+ * \delta_2(x)$$

Now,

$$\begin{aligned} S_1 * S_2(x) &= \{(10x)^+ * \delta_1(x)\} * \{(9x)^+ * \delta_2(x)\} \\ &= \{(10x)^+ * (9x)^+\} * \{\delta_1(x) * \delta_2(x)\} \\ &= \{(10x)^+ \wedge (9x)^+\} * \{\delta_3(x)\} \\ &= (9x)^+ * \delta_3(x) \\ &= 9(x - 3)^+ \end{aligned}$$

The second equality resulted from the associativity property of the convolution. The third equality resulted since both  $(10x)^+$  and  $(9x)^+$  are *causal concave* (in the sense defined in the class notes) processes.