Serial Concatenated Trellis Coded Modulation with Inner Rate-1 Accumulate Code

Hugo M. Tullberg, Paul H. Siegel

March 2, 2001

Abstract

In order to combine the extraordinary performance of turbo codes with the bandwidth efficiency of Trellis-Coded Modulation (TCM), several Turbo-TCM schemes have been proposed. Both systems using Parallel Concatenated TCM (PCTCM) and Serial Concatenated TCM (SCTCM) have been shown to achieve good performance. Inspired by the analytical tractability of Repeat-Accumulate (RA) codes, we propose an SCTCM system with an inner accumulate code.

The system consists of simple building blocks; an outer encoder, an interleaver, an inner rate-1/1 code and a mapping onto a Gray-labeled constellation. At the receiver, bit metrics are computed (non-iteratively) and then the inner and outer codes are iteratively decoded by two Soft-Input Soft-Output (SISO) modules separated by appropriate interleavers.

The use of an inner accumulate has both theoretical and practical advantages. We derive coding theorems, stating that when the signal-to-noise ratio (SNR) exceeds a certain, system-specific, threshold, the bit error probability goes to zero as the block-length goes to infinity. For finite (short) block-lengths, we derive a new bound, which is an improvement over the union bound.

Since the inner code is rate-1/1, the spectral efficiency of this system is determined by the outer code and constellation size. The outer code can easily be tailored to the desired rate, possibly by puncturing, and we do not have to find high rate outer and inner convolutional codes.

1 System Description

The encoder consists of an outer constituent code, an interleaver, an inner constituent code and a memoryless mapper, as shown in Figure 1. The outer code is a block code, either formed by terminating a rate $r_c = k/n$ convolutional code or by concatenating several short block codes, such as $r_c = k/(k+1)$ Parity check codes or $r_c = 1/q$ Repeat codes.

The information bit sequence u is fed to the outer encoder, which outputs the encoded bit sequence v. The interleaver is a random or S-random interleaver of size N and acts on all bits in a block. The inner code is a $r_c = 1/1$ Accumulate code. The output of the Accumulator is a running modulo-2 sum of the inputs. The memoryless

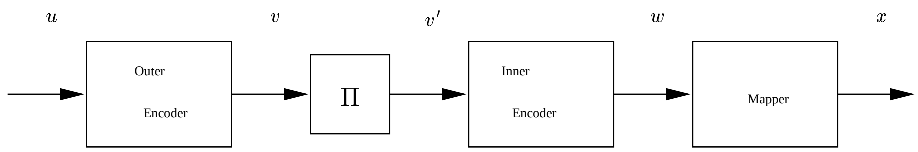

Figure 1: Basic encoder structure

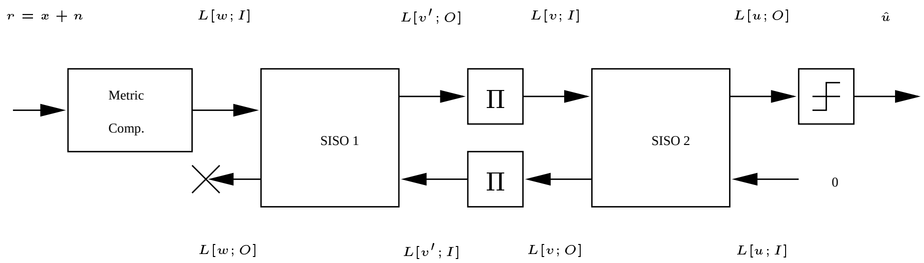

Figure 2: Basic decoder structure

mapper maps an m-tuple of bits to a constellation point x in a Gray-labeled constellation $\mathcal{X}$ of size $|\mathcal{X}| = M = 2^m$. Note that m is not necessarily equal to n. The Gray labeling allows for a simple relationship between Hamming distance d_H and squared Euclidean distance d_E^2.

The decoder consists of a bit metric calculator, a soft-input soft-output (SISO) module matched to the inner code and an SISO module matched to the outer code, separated by appropriate interleavers and deinterleavers. The final, hard binary decision is given by a slicer. The decoder is depicted in Figure 2.

In the basic version of the proposed SCTCM scheme, we do not iterate over the metric computation. The a posteriori probability $\Pr[x|r]$ can be expressed as $\Pr[x|r] = p[r|x]\Pr[x]/p[r]$ by Bayes' rule. If we assume that all constellation points are transmitted equally often, we have $\Pr[x|r] \propto \Pr[r|x]$. The decoding metric is $\lambda_b^i = \log \sum_{x \in \mathcal{X}_b^i} \Pr[r|x]$, where $\mathcal{X}_b^i = \left\{ x \in \mathcal{X} | \ell^i = b \right\}, b = \{0, 1\}, i = \{1, \ldots, m\}$ is the set of constellation points in the constellation $\mathcal{X}$ such that the i-th bit in the binary label of the points has the value b. In Gaussian noise and for equal energy constellations, we get

$$\lambda_b^i = \log \sum_{x \in \mathcal{X}_b^i} e^{\frac{r_1 x_1 + r_2 x_2}{\sigma^2}} \tag{1}$$

The input to the SISO is then formed as $L[w; I] = \lambda_b^i - \lambda_{\bar{b}}^i$.

This system exhibits a threshold in the decoding, i.e., the performance is bad until a certain SNR is reached when the improves dramatically. This threshold becomes more and more pronounced as the block-length increases.

For convolutional codes with $d_{\text{free}}^{(o)} > 2$, errors arise when the iterative decoder fails to converge, rather than converging to an erroneous codeword. However, for parity check codes we have observed convergence to erroneous codewords at small Hamming distances from the transmitted codeword.

We have also generalized the system to have multiple Accumulate codes (separated by interleavers) and to include iterative demodulation. For an outer Parity check code, multiple Accumulate codes are necessary to achieve good word error performance.

2 Performance Analysis

The use of a Gray-labeled constellation allows us to divide the system into two parts; a concatenation of binary codes and a mapping from Hamming distance to squared Euclidean distance (SED). This allows us to use a simplified analysis based on the Union Bound. For a memoryless channel and Maximum Likelihood (ML) decoding, the word and bit error probabilities are given by the union bound

$$P_w \leqslant \sum_{h=1}^{n} \sum_{w=1}^{k} A_{w,h} z^h = \sum_{h=1}^{n} A_h z^h \qquad P_b \leqslant \sum_{h=1}^{n} \sum_{w=1}^{k} \frac{w}{k} A_{w,h} z^h \tag{2}$$

where $A_{w,h}$ are the coefficients in the Input Output Weight Enumerator (IOWE) $A(W,H) = \sum_{w,h} A_{w,h} W^w H^h$ and z^h is an upper bound on the bitwise error probability.

For serial concatenation of two codes through a uniform interleaver, the IOWE is given by

$$A_{w,h}^{(c)} = \sum_{h_o=0}^{N} \frac{A_{w,h_o}^{(o)} A_{h_o,h}^{(i)}}{\binom{N}{h_o}} \tag{3}$$

where $A_{w,h_o}^{(o)}$ $A_{h_o,h}^{(i)}$ are coefficients from the IOWEs for the outer and inner codes.

For the *binary-input* AWGN channel, the Bhattacharayya bound gives $z = e^{-r_c E_b/N_0}$, where r_c is the code rate and E_b/N_0 is the energy per bit to noise ratio. For a Gray-labeled constellation, we can upper bound z as $z \leqslant e^{-\frac{r_c m E_b}{4N_0} d_{\min}^2}$, where d_{min}^2 is the minimum SED in the constellation. This expression assumes a linear relationship between Hamming distance and SED, which usually is not the case for higher order constellations and the bound may be loose.

3 Improved Bound

We derive a new bound which is an improvement over the union bound. To this end, we

- Count the number of runs of 1's in a codeword.

- Find the distribution of runlengths, given an output weight h and t runs.
- Find tighter expressions for the SED, given a run of 1's of weight h_i.
- Note that the IOWE can be expurgated.

To count the number of runs, we augment the IOWE for the serial concatenation with a “run counter”, denoted T. By properties of the Accumulate code, codewords with t runs must have an intermediate weight h_o of $h_o = 2t$ or $h_o = 2t - 1$, and we get the augmented IOWE as

$$A^{(c)}(W,H,T) = \sum_{w,h} \sum_{h_o}^{N} \frac{A^{(o)}_{w,h_o} A^{(i)}_{h_o,h}}{\binom{N}{h_o}} W^w H^h T^{\lceil \frac{h_o}{2} \rceil} \quad (4)$$

The probability of a run of length j, given a codeword of weight h consisting of t runs, is

$$\Pr[j|h,t] = \frac{\binom{h-1-t}{t-1} - \binom{h-2-t}{t-1}}{\binom{h-1}{t-1}}, \quad 0 \leqslant j \leqslant h-t \quad (5)$$

The SED depends on the constellation and labeling. As an example, in Table 1 we list the possible distances for runs of length 1 through 6 for an 8-PSK constellation. Note that for runs of length 5 and greater, the patterns repeat with the addition of one or more symbols at distance 3.

$$P_b \leqslant \sum_{h=1}^{n} \sum_{w=1}^{k} \sum_{t=1}^{\lceil \frac{h}{2} \rceil} \frac{w}{k} A_{w,h,t} f(\mathcal{X}, h, t)$$

where $A_{w,h,t}$ is the augmented IOWEF and $f(\mathcal{X}, h, t)$ is a distance function that depends on the constellation $\mathcal{X}$ and the weight distribution of the codeword, as discussed above.

The lower bound on the distance gives us an upper bound on the performance (Figure 3, denoted “IB exp.”). Though not a bound, the average distance may better reflect the actual performance of the code (Figure 4, denoted “IB ave.”).

4 Coding Theorems

For the described SCTCM system we can derive coding theorems for the Word Error Probability, P_w, and Bit Error Probability, P_b:

- For an outer repeat code of rate $r_c \leqslant 1/3$, there exist a z^* such that for $z < z^*$, $P_W \to 0$ as $N \to \infty$.
- For an outer parity check code, $d^o_{\text{free}} = 2$, and a single inner accumulate code, there exist a z^* such that for $z < z^*$, $P_b \to 0$ as $N \to \infty$. $P_W \nrightarrow 0$ as $N \to \infty$.

Runlength	Patterns	SED	Lower bound	Average
1	001	0.5858		
	010	3.4142	0.5858	1.5286
	100	0.5858		
2	001,100	$2 \cdot 0.5858$		
	011	4	1.1716	2.8619
	110	2		
3	001,110	$0.5858 + 2$		
	011,100	$4 + 0.5858$	2.5858	4.0000
	111	3.4142		
4	001,111	$0.5858 + 3.4142$		
	011,110	$4 + 2$	4.000	5.1381
	111,100	$3.414 + 0.5858$		
5	001,111,100	$2 \cdot 0.5858 + 3.4142$		
	011,111	$4 + 3.4142$	4.5858	6.2761
	111,110	$3.4142 + 2$		
6	001,111,110	$0.5858 + 3.4142 + 2$		
	011,111,100	4+3.4142+0.5858	6.0000	7.4142
	111,111	$2 \cdot 3.4142$		

Table 1: Patterns and distances for runs of length 1 through 6 for an 8-PSK Gray-labeled constellation. SED to '000'-point, minimum and average distances are taken over all constellation points.

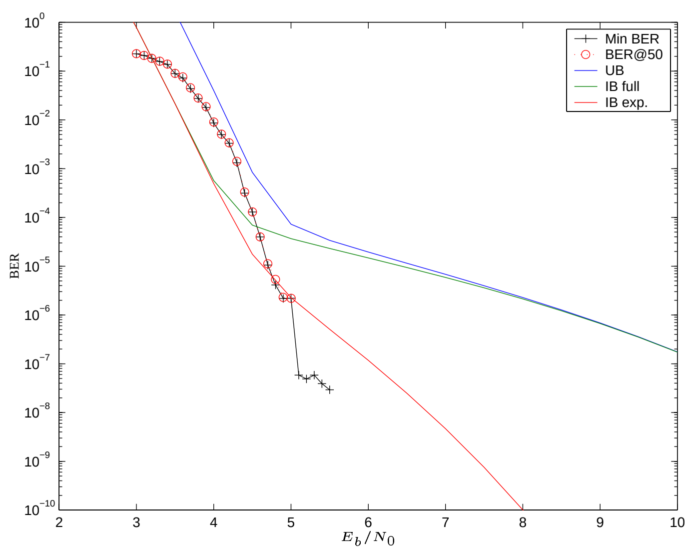

Figure 3: Improved bound, $N = 1536$, mem $= 2$

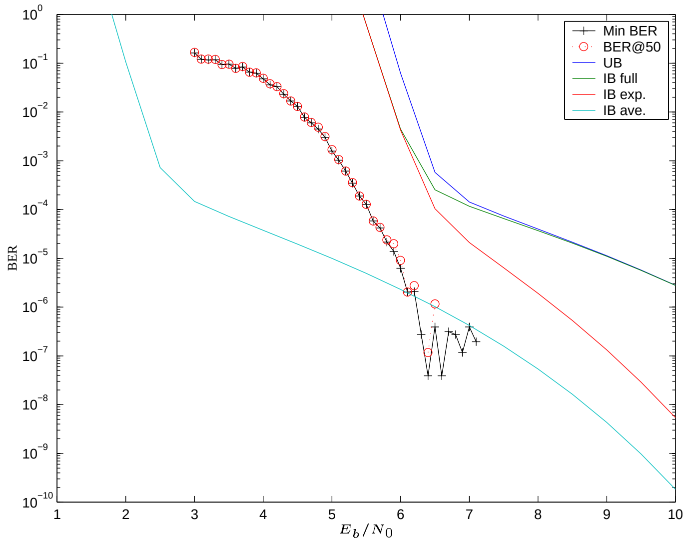

Figure 4: Improved bound, $N = 384$, mem $= 2$

- For an outer parity check code, and two inner accumulate codes, there exist a z^* such that for $z < z^*$, $P_W \to 0$ as $N \to \infty$.

The proof is an extension of the technique of Divsalar, et al. [1]. The idea is to express the union bound in the form

$$P_W \leqslant \sum_h e^{h(F(\cdot)+\ln z)}$$

where $F(\cdot)$ depends on the outer and inner codes and z is the channel parameter. Then, if $\ln z < -F(\cdot)$, $P_W \to 0$ as $N \to \infty$. The expression for P_b is similar.

For parity check codes and repeat codes, we can express the Weight Enumerator (WE) in closed form and compute numerical values for the thresholds.

4.1 Fading Channels

When we consider a fading channel, we get a different channel parameter z. For independent Rayleigh fading, the fading power α is exponentially distributed. For BPSK modulation, the conditional channel parameter is $z_{\bar{\alpha}}^h = e^{-r\bar{\alpha}E_b/N_0}$ where $\bar{\alpha}$ is a vector of fading values for the h positions where the codewords differ. By integrating over the fading power density we get the unconditional channel parameter for BPSK as

$$z^h = \left[\frac{1}{1 + rE_b/N_0}\right]^h \tag{6}$$

and, similarly, for higher order constellations

$$z^h = \left[\frac{1}{1 + \frac{rmd^2_{\min}}{4} \frac{E_b}{N_0}} \right]^h \tag{7}$$

4.2 Numerical Results

In the table below, we list the E_b/N_0 thresholds for systems using a $r_c = 1/3$ Repeat-Accumulate (RA), $r_c = 2/3$ Parity-Accumulate (PA) and $r_c = 2/3$ Parity-Accumulate-Accumulate (PAA) codes. Note that for PA, the WER does not go to zero with increasing block-lengths, and the thresholds are for the BER.

	RA (WER)		PA (BER)		PAA (WER)	
	BPSK	8-PSK	BPSK	8-PSK	BPSK	8-PSK
AWGN	2.20	5.77	7.78	11.35	10.79	14.46
Rayleigh	3.46	7.03	19.05	22.62	36.50	40.07

Table 2: Thresholds for RA, PA and PAA systems.

In plots below we compare the thresholds to simulation results. In Figure 5, we show a system using RA codes and 8-PSK modulation. There is a substantial discrepancy between the thresholds and simulation results, since the threshold computation is based on the union bound and since the bound on z is loose. In Figure 6, we show PA and PAA codes with 8-PSK modulation over an AWGN channel. In this case the thresholds are so loose, they only serve as existence proofs. We notice that P_W does not go to zero for PA codes, but for PAA codes no word errors occurred for $E_b/N_0 > 6$ dB.

5 Simulation Results

5.1 AWGN

In Figure 7 we show the performance in AWGN and compare to the SCTCM scheme reported in [2]. Both systems use an outer $r_c = 3/4$, memory-2 convolutional code and 16-QAM modulation, and the spectral efficiency is 3 bits/s/Hz. The block length is 12288 information symbols, corresponding to 4096 channel symbols. Our system use a simpler inner code, but still achieve the same performance. After 20 iterations we have not seen errors for $E_b/N_0 > 5.4$ dB.

5.2 Fading

Here we show the performance over a correlated, flat Rayleigh fading channel. We have simulated our system as well as BICM-ID [3]. The outer code is a $r_c = 2/3$,

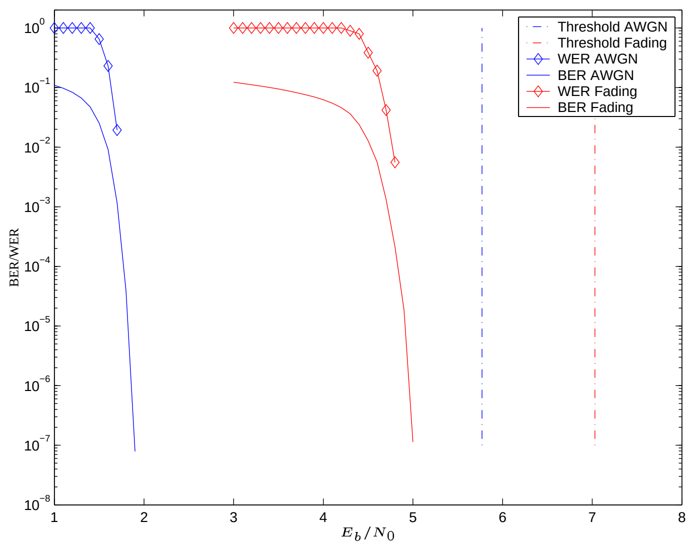

Figure 5: Thresholds and simulation results for RA codes with 8-PSK modulation over AWGN and Rayleigh fading channels. Interleaver length $N = 49152$. Some residual word errors are indicated for higher E_b/N_0-values.

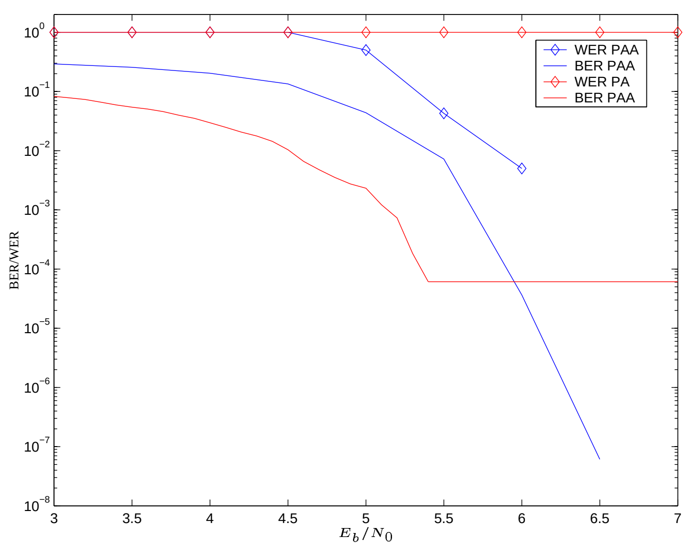

Figure 6: Simulation results for PA and PAA codes with 8-PSK modulation over an AWGN channel. $N = 24576$

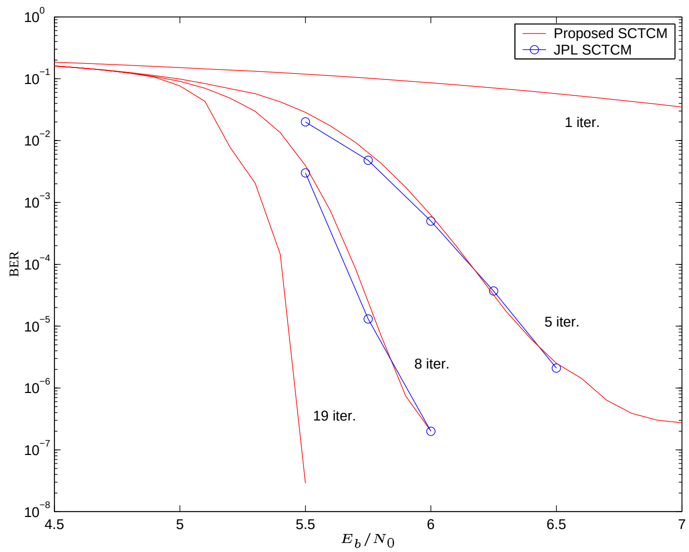

Figure 7: Performance comparison in AWGN

memory-3 convolutional code and the modulation is 8-PSK. The information block-length is 16384 bits, corresponding to 8192 channel symbols. The performance after 1, 5, and 11 iterations is shown. Even though BICM was initially deviced to increase diversity over a fading channel, we see that our system compare favorably.

5.3 Block-length

In Figure 9 we show the effect of the block-length. All systems use an outer convolutional code with free distance $d_{\text{free}}^{(o)} = 4$ and 8-PSK modulation. The information block-lengths are 2k, 8k and 128k bits. BER curves are shown for 12 iterations and WER curves after at most 20 iterations. Note that for the 128k block-length, no word errors occur for $E_b/N_0 > 3.8$ dB.

6 Conclusions

- We propose an SCTCM system with very simple building blocks. Simulations show performance comparable to or better than more complex systems.

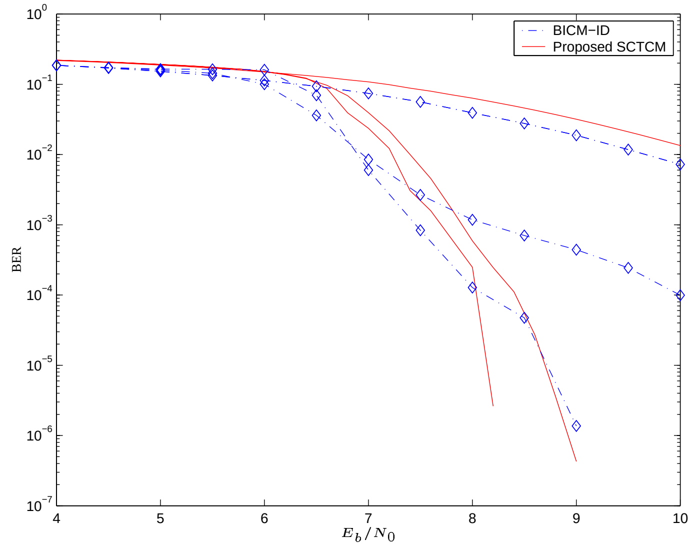

Figure 8: Comparison in Rayleigh fading.

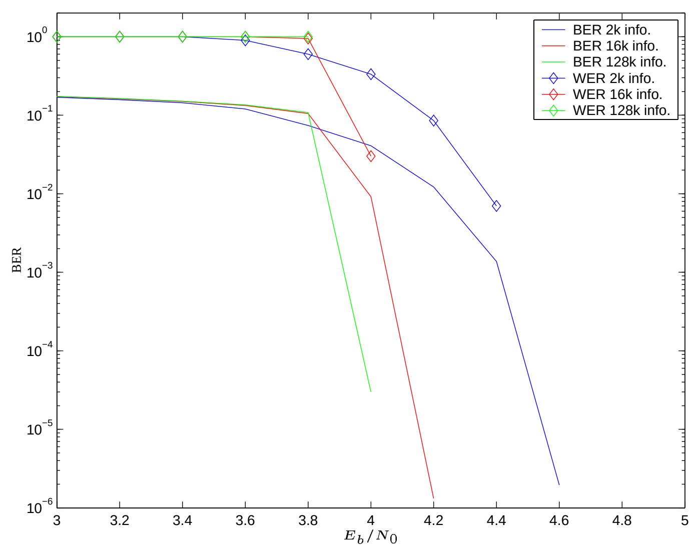

Figure 9: Effect of the block length

- We can show coding theorems for the proposed system, stating that the word or bit error probability will go to zero as the block-length goes to infinity.
- For finite block-length, we derive a new improved bound on the performance.

References

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